Odd potentials in supersymmetric quantum mechanics

1990 J. Phys. A: Math. Gen. 23 L659

(http://iopscience.iop.org/0305-4470/23/13/005)

J. Phys. A: Math. Gen. 23 (1990) L659-L663. Printed in the UK

LETTER TO THE EDITOR

Odd potentials in supersymmetric quantum mechanics

Cao Xuan Chuan†

Observatoire de Nice, BP 139, Nice Cedex, France

Received 23 April 1990

Abstract. We investigate the case of odd potentials for which supersymmetry is broken. It is, however, always possible to construct a representation which includes a non-degenerate zero-energy ground state. This method provides a convenient means to generate a wide variety of physical situations which are related to supersymmetry.

It is well known that supersymmetry is broken if there exists no zero-energy ground state or there is degeneracy of this ground state (Witten 1981, Gedenshtein and Krive 1985, Berstein and Brown 1983, Jaffe *et al* 1987). For the simplest case of a two-component theory ϕ_1, ϕ_2 in one-dimensional space, this problem can be analysed with the following system of two coupled differential equations of first order (Salomonson and Van Holten 1982, Cooper and Freedman 1983):

$$\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{d}x}-v'\right)\phi_1 &= -(2E)^{1/2}\phi_2 \\ \left(\frac{\mathrm{d}}{\mathrm{d}x}+v'\right)\phi_2 &= (2E)^{1/2}\phi_1 \end{aligned} \tag{1}$$

where $v' = \mathrm{d}v/\mathrm{d}x$, $v(x)$ is the superpotential which we assume to be of the Witten form ($v(x)\to\infty$ as $|x|\to\infty$) and E is the energy.

For zero energy ($E=0$), this system becomes uncoupled and the breaking of symmetry will depend on the behaviour of this potential at infinity, that is to say, it must be an asymptotically even function of x ($v(x)=v(-x)$ as $x\to\pm\infty$).

On the other hand, asymptotically odd potentials ($v(x)=-v(-x)$ as $|x|\to\infty$) which lead to the breaking of symmetry are usually ignored in conventional theories, mostly because of the lack of normalisibility of both components ϕ_1, ϕ_2. This exclusion is, however, not necessary as can be seen below where the proof of the following statement is given.

Statement. For asymptotically odd potentials, it is always possible to construct a representation which includes a non-degenerate zero-energy ground state.

A number of examples and applications will be discussed as illustrations and consequences of this statement.

† Permanent address: 01 Parvis de Breuil, 92160 Antony, France.

0305-4470/90/130659+05\$03.50

Proof. Let the new representation (ψ_1, ψ_2) be related to the former one (ϕ_1, ϕ_2) by

$$\psi = T\phi \tag{2}$$

where the 2×2 transformation matrix T is

$$T = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \tag{3}$$

$c = c(x)$ being assumed to be an analytical function of x. Note that the inverse T^{-1} of T is simply

$$T^{-1} = \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix}$$

and $\det|T| = 1$.

We then have:

$$T\left[I\frac{\mathrm{d}}{\mathrm{d}x} + A\right]T^{-1}\psi = 0 \qquad A = \begin{pmatrix} -v^{-1} & (2E)^{1/2} \\ -(2E)^{1/2} & v' \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or more explicitly

$$\begin{aligned} \left[\frac{\mathrm{d}}{\mathrm{d}x} - \bar{v}'\right]\psi_1 &= -c^2(2E)^{1/2}\psi_2 \\ \left[\frac{\mathrm{d}}{\mathrm{d}x} + \bar{v}'\right]\psi_2 &= +\frac{1}{c^2}(2E)^{1/2}\psi_2 \end{aligned} \tag{4}$$

in which

$$\bar{v}' = v' + \frac{\mathrm{d}}{\mathrm{d}x}\log c(x). \tag{5}$$

For $E = 0$, the RHS is zero. As $v(x)$ is assumed odd, and as the two components ϕ_1, ϕ_2 are not determined in this case, we have to choose $c(x)$ such that at least one of the two components ψ_1, ψ_2 is normalisable. Many choices are obviously possible but for the moment we shall consider the simplest one

$$c(x) = \exp(-v(x) + v^2(x)) \tag{6}$$

that is to say, $\bar{v} = +v^2$. The asymptotic forms of $\psi_{1,2}$ are then $\psi_{1,2} \simeq \mathrm{e}^{\pm v^2}$, i.e. ψ_2 is normalisable but not ψ_1. With the inverse transformation, $\phi = T^{-1}\psi$, it can be verified that $\phi_{1,2} \sim \mathrm{e}^{\pm v}$ as expected.

Relation to supersymmetry. Define the new supergenerators $\bar{Q}^{\pm}$ as

$$\begin{aligned} &\bar{Q}^+ = \begin{pmatrix} 0 & \bar{A}^+ \\ 0 & 0 \end{pmatrix} \qquad \bar{Q}^- = \begin{pmatrix} 0 & 0 \\ \bar{A}^- & 0 \end{pmatrix} \\ &\bar{A}^{\pm} = \pm\frac{\mathrm{d}}{\mathrm{d}x} + \bar{v}' \end{aligned} \tag{7}$$

so that

$$[\bar{A}^+, \bar{A}^-] = 2\bar{v}'' \qquad \bar{Q}^+ = \bar{Q}^- = 0$$

and the Hamiltonian $\bar{H}_+$ and its partner $\bar{H}_-$ are

$$\{\bar{Q}^+, \bar{Q}^-\} = 2\bar{H}_{\mp} \qquad \bar{H} = \begin{pmatrix} \bar{H}_+ & 0 \\ 0 & \bar{H}_- \end{pmatrix}. \tag{8}$$

Note that $\bar{Q}^{\pm}$ are conserved quantities because $[\bar{Q}^{\pm}, \bar{H}]=0$, where $\{\,,\}$ and $[\,,]$ mean commutator and anticommutator. The corresponding potentials are then

$$\bar{V}_{\pm}=\bar{v}'^2 \pm \bar{v}''. \tag{9}$$

Interpretation. It is clear that the components ψ_1, ψ_2 can be considered as the bosonic and fermionic components in this new representation corresponding to a new superpotential $\bar{v}$ which, with our choice, is simply $-v^2$. From (6) we find

$$\bar{V}_{\pm}=4v^2v'^2 \pm 2(v'^2+vv''). \tag{10}$$

Denote the energy spectrum $\{\bar{E}_n\}$ corresponding to $\bar{V}_{\pm}$ and $\{E_n\}$ to $V_{\pm}=v'^2 \pm v''$. Obviously these two spectra are different because E_n are not eigenvalues in the (ψ) representation except for special cases to be seen below. $\{\bar{E}_n\}$ has a non-degenerate zero-energy ground state so that its Witten index $\Delta_w \neq 0$.

Examples. (*a*) Let $v(x)$ be linear in the sense $v(x)=2^{-1/2}x$; together with (10) the corresponding Hamiltonian in the ψ representation is:

$$\bar{H}_{\pm}=-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\tfrac{1}{2}[x^2 \mp 1] \tag{11}$$

which can be associated with the problem of an electron moving in a magnetic field with a gyromagnetic ratio $g=2$ (Gedenshtein and Krive 1983, Haymaker and Rau 1988).

Consider now any potential $v(x)$ and choose for $c(x)$ the form

$$c(x)=\mathrm{e}^{-v(x)}\cosh^n v(x) \qquad n>0. \tag{12}$$

Then

$$\psi_2 \sim (\cosh^n v)^{-1} \qquad \psi_1 \sim \cosh^n v$$

i.e. ψ_2 can be normalised but not ψ_1. Using (9) we obtain

$$\bar{V}_{\pm}=nv'^2[n-(n\mp 1)\operatorname{sech}^2 v(x)] \pm nv'' \tanh v(x)$$

which is fairly complicated in the general case.

(*b*) For the special case $v(x)=x$, $n=1$, we find

$$\bar{V}_{\pm}=\begin{cases}1\\ 1-2\operatorname{sech}^2 x\end{cases} \tag{13}$$

which is precisely the case discussed in Kwong and Rosner (1986) where it has been shown that if one of the two potentials is constant, its partner must be an even and reflectionless potential (see also Alkhoury and Comtet (1984) where the quantity ψ_0 should be $N(\cosh x)^{-1}$.

(*c*) For the more general case $v(x)=\alpha x$, $n=1$ we may also define the potential $\bar{\bar{V}}=\bar{V}-n^2\alpha^2$; then ($\alpha$: parameter):

$$\bar{\bar{V}}_{\pm}=-2\alpha^2\operatorname{sech}^2 \alpha x \tag{14}$$

which can be regarded as an instantaneous soliton of the KdV equation. Note also that here $\bar{v}'=\alpha\tanh\alpha x$ which is the Pösch–Teller case discussed by Cooper *et al* (1988).

From (a), (b) and (c) it is interesting to note that starting from the same type of linear potential but with different choices of $c(x)$, we have generated three apparently unrelated situations which have been formulated by conventional methods with three different superpotentials.

Note also the relative simplicity of the solution of the Schrödinger equations for the zero-energy ground state which may be useful if we wish to extend to the factorisation procedures (Sukumar 1985, Andrianov *et al* 1984) when $v(x)$ has a complicated structure. Take, for example, the polynomial form

$$v(x)=\sum_{m=0}^{N} a_m x^m \tag{15}$$

for which the Schrödinger equation with the corresponding potential $\bar{V}_\pm$ is very difficult to solve while the zero-energy state wavefunction is simple $\psi_0 = Ae^{-v^2}$, A being the normalisation constant. More precisely, assume now that all coefficients a_m are zero except the one defined by $a_N = a_{n+1} = [2(n+1)]^{-1/2}$. The corresponding quantity $\bar{V}_\pm$ is

$$\bar{V}_\pm = x^{4n+2} \pm (2n+1)x^{2n} \tag{16}$$

which, up to a constant, is exactly the same potential considered in Khare (1985). Although it has a double-well structure with two degenerate minima (the instanton case) supersymmetry nevertheless remains unbroken ($\Delta_w \neq 0$) because its ground state is well defined and normalisable

$$\psi_0 \sim \exp\left[-\frac{x^{2n+2}}{2(n+1)}\right].$$

Note that in this special case we have an identity between the two spectra $\{\bar{E}_n\}$ and $\{E_n\}$ if N is odd (or n even).

On the other hand, the present approach can also be extended to other types of potential. For example, in the Coulomb case, we have $v(x) = -(l+1)\log x$, $x[0, \omega]$; the appropriate choice of $c(x)$ must be $c(x) = \exp[-x/2(l+1)]$. From (9) we obtain

$$\bar{V}_\pm = \frac{L_\pm}{x^2} - \frac{1}{x} + \frac{1}{4(l+1)^2} \tag{17}$$

where $L_\pm$ have the characteristic forms $L_+ = (l+1)(l+2)$ and $L_- = l(l+1)$ which provide a supersymmetric interpretation of the 'accidental degeneracy' in hydrogenic atoms (Kostelecky and Nieto 1984, Haymaker 1986, Lahiri *et al* 1989).

From a more general point of view, the flexibility of the present approach can be appreciated according to the following remarks.

(i) As was stated above, many choices of $c(x)$ are obviously possible. Each choice leads to a modification of the ladder operator but always preserves the supersymmetry character. For example, let $U(x)$, $k(x)$ be arbitrary functions and y a parameter. If we choose for $c(x)$ the form

$$c(x) = \exp\left(-v(x) - \int \frac{U(x)}{k(x)}\,\mathrm{d}x + \mathrm{i}\frac{\partial}{\partial y}\log k(x)\right) \tag{18}$$

then it can be verified easily that the 'ladder operator' $\bar{A}^+$ defined in (7) is in exact agreement with the results from Lahiri *et al* (1988).

(ii) It is clear that $\bar{v}(x)$ can be defined up to a constant λ_1 which is a parameter of the first kind. For example, if we set $c(x) \to e^{\lambda_1}c(x)$, the spectrum $\{\bar{E}_n\}$ will remain unchanged. There is, however, a second kind of parameter λ_c which is directly related to $\tilde{v}'(x)$ and which can be used to define a family of isospectral potentials (Khare and Sukhatme 1989). From this point of view we may note the similarity between the role of the quantity c'/c of present work and the auxiliary function $\phi(x)$ in this reference. For even superpotentials $v(x)$ we have in fact obtained the same conclusions (unbroken symmetry) but the question of odd potentials still remains open.

To summarise, we may conclude that with these few examples above, it will not be unreasonable to suggest that the present approach, with adequate choices of the function $c(x)$, can in fact provide a convenient method to generate a broad spectrum of physical situations which are related to supersymmetry.

References

Akhoury R and Comtet A 1984 *Nucl. Phys.* B **246** 253
Andrianov A A, Borisov N V and Ioffe M V 1984 *Phys. Lett.* **105A** 19
Berstein M and Brown L S 1983 *Phys. Rev. Lett.* **129B** 432
Cooper F and Freedman B 1983 *Ann. Phys.* **146** 262
Cooper F, Ginnocchio J N and Wipf A 1988 *Phys. Lett.* **129A** 3 145
Gedenshtein L E and Krive I V 1985 *Usp. Fiz. Nauk* **146** 4
Haymaker R W and Rau A R P 1986 *Am. J. Phys.* **54** 928
Jaffe A, Lesniewski A and Lewenstein M 1987 *Ann. Phys.* **178** 313
Khare A 1985 *Phys. Lett.* **161B** 131
Khare A and Sukhatme V 1989 *J. Phys. A: Math. Gen.* **22** 2847
Kostelecky V A and Nieto M N 1984 *Phys. Rev. Lett.* **53** 2285
Kwong W and Rosner J D 1986 *Prog. Theor. Phys.* **86** 509
Lahiri A, Roy P K and Bagchi B 1988 *Nuovo Cimento* **100A** 5 797; 1989 *Int. J. Theor. Phys.* **28** 183
Salomonson P and Van Holten J M 1982 *Nucl. Phys. B* **196** 509
Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2917
Witten E 1981 *Nucl. Phys.* B **185** 513